

# Wall-crossing of the motivic Donaldson-Thomas invariants

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## Abstract

We study motivic Donaldson-Thomas invariants in the sense of [BBS]. A wall-crossing formula under a mutation is proved for a certain class of quivers with potentials. The class includes the quivers for the canonical bundles of del-Pezzo surfaces with geometric helices. The formula is the same as [KSb, KSa] and is described by quantum dilogarithms.

## Introduction

In this article we study the *motivic Donaldson-Thomas* (DT in short) *invariants* introduced in [KSb, BBS].

The DT invariant for a Calabi-Yau 3-fold  $Y$  is a counting invariant of coherent sheaves on  $Y$ , which it is introduced in [Tho00] as a holomorphic analogue of the Casson invariant on a real 3-manifold. The moduli space involves a symmetric obstruction theory and a virtual fundamental cycle [BF97, BF08]. The invariant is defined as a integration of the constant function 1 over the virtual fundamental cycle.

The DT invariant has the other description : it coincides with the weighted Euler characteristic weighted by the Behrend function. It is known that the moduli space of coherent sheaves on  $Y$  can be locally described as the critical locus of a function which is called a *holomorphic Chern-Simons functional* (see [JS]). The value of the Behrend function is given by the Euler characteristic of the *Milnor fiber* of the Chern-Simons functional [Beh09].

Following these results of Behrend, it is proposed in [KSb, BBS] to study *motivic Milnor fiber* as a motivic version of the DT invariant so that we can

get a refinement of the ordinary DT invariant by applying a suitable cohomology functor for the motivic one. Such a refinement has been expected in string theory [IKV09, DG, DGS11].

In [KSb], Kontsevich and Soibelman provided a wall-crossing formula for motivic DT invariants up to a certain identity for motivic Milnor fibers ([KSb, Conjecture 4]). The aim of this article is to give an alternative proof of the wall-crossing formula for ([BBS]'s) motivic DT invariants in a spacial setting.

## Main result

Let  $(Q, W)$  be a quiver with a potential (QP in short). In this paper, we assume that  $W$  is finite, i.e. a finite linear combination of oriented cycles.

Let  $\mathcal{D}_{Q,W}$  be the derived category of dg modules with finite dimensional cohomologies over the (non-complete) Ginzburg's dg algebra and  $\text{mod} J(Q, W)$  be the category of finite dimensional modules over the (non-complete) Jacobi algebra which is the core of the natural bounded t-structure of  $\mathcal{D}_{Q,W}$ . The moduli stack of objects is canonically described as the critical locus of a function  $f_W$  on a smooth stack  $\mathfrak{M}_Q$ . We call the function as the *Chern-Simons functional*. We define the motivic DT invariant by the virtual motive  $[\text{crit}(f_W)]_{\text{vir}}$  (Definition 1.6) of the critical locus of the Chern-Simons functional<sup>1</sup>.

For a vertex  $k$  without loops, let  $\mu_k(Q, W) = (Q', W')$  be the mutation in the sense of [DWZ08]. We assume that  $Q'$  is the quiver mutation in the sense of Fomin-Zelevinsky and  $W'$  is finite<sup>2</sup>. Keller-Yang showed that  $\mathcal{D}_{Q,W}$  and  $\mathcal{D}_{Q',W'}$  are equivalent [KY, Kela]. We want to describe the relation between the motivic DT invariant for  $(Q, W)$  and the one for  $(Q', W')$ .

The main theorem in this paper is the following :

**Theorem 0.1.** *Assume that*

- $(Q, W)$  has a cut (Definition 2.1),
- $k$  is a strict source of  $C$  (Definition 2.6).

*Then we have*

$$\mathcal{A}_{Q',W'} = \mathbb{E}(s_k[1]) \times \mathcal{A}_{Q,W} \times \mathbb{E}(s_k)^{-1}$$

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<sup>1</sup>The smooth stack  $\mathcal{M}_Q$  is described as a quotient stack divided by a special algebraic group  $G$ . Actually we do not work on critical loci for stacks but only for varieties and define motivic invariants by the quotient of the virtual motives by  $G$ .

<sup>2</sup>In [DWZ08], it is shown that if the potential  $W$  is generic then  $Q'$  is the Fomin-Zelevinsky mutation of  $Q$ . Finiteness of  $W$  is stronger assumption.

where  $\mathcal{A}_{Q,W}$  is the generating series of the motivic Donaldson-Thomas invariants and  $\mathbb{E}(s_k)$  is the “motivic dilogarithm” (Example 1.2). This is an equation in the “motivic torus” (Definition 3.10).<sup>3</sup>

By taking the weight polynomial we get the following :

$$A_{Q',W'} = \mathbb{E}_q(s_k[1]) \times A_{Q,W} \times \mathbb{E}_q(s_k)^{-1}$$

where  $A_{Q,W}$  is the generating series of the “refined Donaldson-Thomas invariants” (Definition 3.15) and  $\mathbb{E}_q(s_k)$  is the quantum dilogarithm. This is an equation in the “quantum torus” (Definition 3.14).

## Sketch of the proof

### First step

The first step is to show the *factorization property*. Take a stability condition, then each object  $\text{mod}J(Q, W)$  has the unique Harder-Narashimhan filtration. Types of the Harder-Narashimhan filtrations induce a filtration by open sets on the moduli stack (see §3.2). In particular, each stratum is smooth. Using this filtration we want a formula which describes the generating function of the motivic DT invariants as the product of the generating functions of the motivic invariants of the moduli stacks of semi-stable objects. To get the formula, we need the following ;

Let  $X$  be a smooth stack,  $f$  be a function on  $X$  and  $Y \subset X$  be a smooth substack of codimension  $d$ . Then,

$$[\text{crit}(f)]_{\text{vir}} \stackrel{?}{=} [\text{crit}(f|_{X \setminus Y})]_{\text{vir}} + \mathbb{L}^{-\frac{d}{2}} \cdot [\text{crit}(f|_Y)]_{\text{vir}}. \quad (1)$$

In §3, we assume that we have a cut  $C$  of the QP  $(Q, W)$ , that is, a nonnegative grading

$$g_C: Q_1 \rightarrow \mathbb{Z}_{\geq 0}$$

such that  $W$  is homogeneous of degree 1. Then the moduli stack involves a  $\mathbb{C}^*$ -action so that we can apply [BBS, Theorem B.1] (Theorem 1.4). The equation (1) directly follows [BBS, Theorem B.1] (see Proposition 3.1).

**Remark 0.2.** In [KSa], Kontsevich-Soibelman introduce the cohomological Hall algebra (COHA in short) which provide another realization of a refinement of the DT invariant. The factorization property for the COHA is shown in [KSa, §5]. For the COHA, the Thom isomorphism is the counterpart of the equation (1).

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<sup>3</sup>This is an equation of infinite power series. Since we have to make it clear in which completion we work, I use the equal sign with quotation mark “=”.

Applying the factorization property in our setting, we can describe the generating function of the motivic DT invariants as the product of the generating functions of the motivic invariants of the moduli stacks of objects in

$$\mathcal{S} := \{s_k^{\oplus n} \mid n \geq 0\}$$

and

$${}^\perp\mathcal{S} := \{X \in \text{mod}J(Q, W) \mid \text{Hom}(X, s_k) = 0\}$$

where  $s_k$  is the simple  $J(Q, W)$ -module corresponding to the vertex  $k$ . The generating function for  $\mathcal{S}$  is given by the *quantum dilogarithm*.

Second step

In the same way, we can describe the generating function for  $(Q', W')$  as the product of the generating functions of the motivic invariants of the moduli stacks of objects in

$$(\mathcal{S}')^\perp := \{X \in \text{mod}J(Q', W') \mid \text{Hom}(s'_k, X) = 0\}.$$

and

$$\mathcal{S}' := \{(s'_k)^{\oplus n} \mid n \geq 0\}$$

where  $s'_k$  is the simple  $J(Q', W')$ -module. It is shown in [KY] that the derived equivalence is given by *tilting* with respect to the simple module  $s_k$ , that is, in the derived category we have

$$\mathcal{S}' = \mathcal{S}[1], \quad (\mathcal{S}')^\perp = {}^\perp\mathcal{S} \quad (\text{see Figure 1}).$$

Now, we get two Chern-Simons functionals which realize the moduli stack

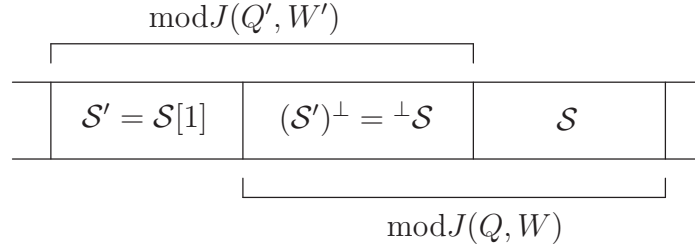


Figure 1:  $\text{mod}J(Q, W)$  and  $\text{mod}J(Q', W')$

of objects in  $(\mathcal{S}')^\perp = {}^\perp\mathcal{S}$  as the critical loci ; one is the restriction of  $f_W$  and the other is the restriction of  $f_{\mu_k W}$ . A priori, the virtual motive depends not only on the scheme structure of the critical locus but also on the choice of the Chern-Simons functional. So we need to show the following :

The virtual motives of the moduli stack of objects in  $(\mathcal{S}')^\perp = {}^\perp\mathcal{S}$  defined by  $f_W$  and  $f_{W'}$  coincide. (2)

Combine (2) with the arguments above, we can describe the relation between the motivic DT invariants for  $(Q, W)$  and for  $(Q', W')$  in terms of the quantum dilogarithm.

We prove (2) under the assumption in Theorem 0.1 (Proposition 4.6). The proof consists of the following two steps :

(A) By taking the torus fixed part of the Jacobi algebra  $J(Q, W)$  we can define the *truncated Jacobi algebra*  $J(Q, W)_C$  (§2.1) and we have the following identity (Theorem 4.6) :

$$\begin{aligned} & \text{The virtual motive of moduli stack of } J(Q, W)\text{-modules} \\ &= \text{the motive of the moduli stack of } J(Q, W)_C\text{-modules.} \end{aligned} \quad (3)$$

This is a generalization of [BBS, Equation (2.4)] and [Hua, Theorem 9.5].

(B) If  $k$  is a *strict source* (Definition 2.6), we can take a cut  $C'$  of  $(Q', W')$  and show an identity between the moduli stack of  $J(Q, W)_C$ -modules and the one of  $J(Q', W')_{C'}$ -modules (Proposition 4.6). This makes us possible to compare the virtual motive of the moduli stack of  $J(Q, W)$ -modules and the one of  $J(Q', W')$ -modules.

## Comments

Let us itemize some applications, related topics and further directions. Some of them will appear in the forthcoming paper.

- (a) Applying Theorem 3.13 for a product of two simply laced Dynkin quivers, we can show a quantized version of dilogarithm identity in conformal field theory [Nak]. More general identities has been already shown by B. Keller [Kelb].
- (b) In [Nag], the author studied cluster algebras by using the ideas in Donaldson-Thomas theory. It is expected that we can study quantum cluster algebras [BZ06] using motivic Donaldson-Thomas theory.
- (c) In this paper, the result of Behrend-Bryan-Szendroi [BBS, Theorem B.1] plays a crucial role and existence of desirable torus action is indispensable. We want to show the same results for any generic QP in the future. Once we get (1) and (2), then we can prove the same results immediately.
- (d) During preparing this paper, the author was informed by Sergey Mozgovoy of his related work. In [Moz] he shows a similar result to Theorem 3.13 over finite fields. He uses the result of M. Reineke [Rei].

- (e) During preparing this paper, the author was informed also by Balazs Szendroi of his related work. In [SM], Szendroi and A. Morrison provide a motivic version of the result of [NN]. As a result they realize the refined topological vertex of the generating function of motivic invariants, which has already discussed in physics ([DG, DGS11]). We can apply Theorem 3.13 to study the wall-crossing phenomenon.<sup>4</sup>

## Acknowledgement

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## 1 Motivic Donaldson-Thomas invariants

### 1.1 Motivic ring

Let  $K_0(\text{Var}/\mathbb{C})$  denote the free abelian group on isomorphism classes of complex varieties, modulo relations

$$[X] = [Z] + [U]$$

for  $Z \subset X$  a closed subvariety with complementary open subvariety  $U$ . We can equip  $K_0(\text{Var}/\mathbb{C})$  with the structure of a commutative ring by setting

$$[X] \cdot [Y] = [X \times Y].$$

We write

$$\mathbb{L} = [\mathbb{A}^1] \in K_0(\text{Var}/\mathbb{C})$$

for the class of the affine line. We define the motivic ring

$$\mathcal{M}_{\mathbb{C}} := K_0(\text{Var}/\mathbb{C})[\mathbb{L}^{-1/2}]$$

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<sup>4</sup>The author was informed by Andrew Morrison that he has a generalization for  $\mathbb{C} \times \mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ .

and its localization

$$\widetilde{\mathcal{M}}_{\mathbb{C}} := \mathcal{M}_{\mathbb{C}}[(1 - \mathbb{L}^n)^{-1} : n \geq 1].$$

The following lemma is a consequence of [Bri, Lemma 3.8]:

**Lemma 1.1.** *Let  $X$  (resp.  $Y$ ) be a variety with an action of a special algebraic group  $G$  (resp.  $H$ ). Assume we have an isomorphism of stacks between  $[X/G]$  and  $[Y/H]$ , then we have*

$$\frac{[X]}{[G]} = \frac{[Y]}{[H]} \in \widetilde{\mathcal{M}}_{\mathbb{C}}.$$

Let  $\hat{\mu} := \varprojlim \mathbb{Z}/n\mathbb{Z}$  be the group of roots of unity. We define the Grothendieck group  $K_0^{\hat{\mu}}(\text{Var}/\mathbb{C})$  of varieties with good  $\hat{\mu}$ -actions as in [BBS, §1.4]. We define  $\widetilde{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$  in the same way.

The additive group  $\widetilde{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$  can be endowed with an associative multiplication  $\star$  using convolution involving the classes of Fermat curves [DL98, Loo02]. This product agrees with the ordinary product on the subalgebra  $\widetilde{\mathcal{M}}_{\mathbb{C}} \subset \widetilde{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$  of classes with trivial  $\hat{\mu}$ -actions, but not in general.

## 1.2 Homomorphisms from the motivic ring

Deligne's mixed Hodge structure on compactly supported cohomology of a variety  $X$  gives rise to the  $E$ -polynomial homomorphism

$$E: K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}[x, y]$$

defined on generators by

$$E([X]; x, y) = \sum_{p,q} x^p y^q \sum_i (-1)^i \dim H_{p,q}(H_c^i(X, \mathbb{Q})).$$

This extends to a ring homomorphism

$$E: \widetilde{\mathcal{M}}_{\mathbb{C}} \rightarrow \mathbb{Q}(x^{1/2}, y^{1/2})$$

By the specialization

$$x = y = (xy)^{1/2} = q^{1/2},$$

we get

$$\mathbb{W}: \widetilde{\mathcal{M}}_{\mathbb{C}} \rightarrow \mathbb{Q}(q^{1/2})$$

**Example 1.2.** We put

$$\mathcal{T} := \prod_{n \geq 0} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot e_n$$

where  $e_n$  is a formal variable satisfying  $e_n \cdot e_m = e_{n+m}$ . We put

$$\sum_{n \geq 0} \frac{[\text{pt}]}{[\text{GL}_n] \cdot \mathbb{L}^{-\frac{\dim \text{GL}_n}{2}}} \cdot e_n \in \mathcal{T}.$$

We call this motivic dilogarithm. We extend the homomorphism  $\mathbb{W}$  to

$$\mathcal{T} \rightarrow T := \prod_{n \geq 0} \mathbb{Q}(q^{1/2}) \cdot e_n.$$

Then the image of the motivic dilogarithm under  $\mathbb{W}$  is the quantum dilogarithm ([FK94]) :

$$\sum_{n \geq 0} \frac{q^{n^2/2}}{(q^n - 1) \cdots (q^n - q^{n-1})} e_n \in T.$$

### 1.3 Motivic nearby and vanishing cycles

Let  $f: X \rightarrow \mathbb{C}$  be a regular function on a smooth variety  $X$  and let  $X_0 := f^{-1}(0)$  be the central fiber. Using arc spaces, Denef and Loeser [DL01, Loo02] define the motivic nearby cycle  $[\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$  of  $f$  and the motivic vanishing cycle

$$[\varphi_f] := [\psi_f] - [X_0] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

of  $f$ . Note that if  $f = 0$ , then  $[\psi_0] = -[X]$ .

**Theorem 1.3 (Motivic Thom-Sebastiani Theorem [DL99, Loo02]).** *Let  $f, g$  be regular functions on smooth varieties  $X, Y$ . Then we have*

$$[-\varphi_{f \oplus g}] = [-\varphi_f] \star [-\varphi_g]$$

We say that a  $\mathbb{C}^*$ -action on a variety  $X$  is weakly circle compact if, for all  $x \in X$ , the limit  $\lim_{\lambda \rightarrow 0} \lambda \cdot x$  exists.

**Theorem 1.4 ([BBS, Theorem B.1]).** *Let  $f: X \rightarrow \mathbb{C}$  be a regular morphism on a smooth quasi-projective complex variety. Assume that there exists an action of a connected complex torus  $T$  on  $X$  so that  $f$  is  $T$ -equivariant with respect to a primitive character  $\chi: T \rightarrow \mathbb{C}^*$ , namely  $f(t \cdot x) = \chi(t)f(x)$  for all  $x \in X$  and  $t \in T$ . We further assume that there exists a one parameter subgroup  $\mathbb{C}^* \subset T$  such that the induced action is weakly circle compact. Then the motivic nearby cycle class  $[\psi_f]$  is in  $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$  and is equal to  $[X_1] = [f^{-1}(1)]$ . Consequently the motivic vanishing cycle class  $[\varphi_f]$  is given by  $[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)]$ .*



**Remark 1.5.** In [BBS, Theorem B.1] they assume that, moreover, the fixed point set  $X^{\mathbb{C}^*}$  is compact. As they themselves mention in [BBS, pp15 19-10], this assumption is not necessary.

## 1.4 Virtual motives of critical loci

Let  $f: X \rightarrow \mathbb{C}$  be a regular function on a smooth variety  $X$ , and let  $\text{crit}(f) = \{df = 0\} \subset X$  be its degeneracy locus.

**Definition 1.6.** We define the virtual motive of  $\text{crit}(f)$  to be

$$[\text{crit}(f)]_{\text{vir}} := -\mathbb{L}^{-\frac{\dim X}{2}}[\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\mu}.$$

**Remark 1.7.** The virtual motive may depend not only on the scheme structure of the critical locus but also on the presentation as a critical locus.

We a smooth variety  $X$ , we use the following notation :

$$[X]_{\text{vir}} := [\text{crit}(0: X \rightarrow \mathbb{C})]_{\text{vir}} = \mathbb{L}^{-\frac{\dim X}{2}}[X].$$

## 1.5 Motivic Donaldson-Thomas invariants

Throughout this paper we assume that a quiver is finite and has no loops and oriented 2-cycles. Let  $Q$  be a quiver,  $Q_0$  denote the set of vertices of  $Q$  and  $Q_1$  denote the set of arrows of  $Q$ . For an arrow  $e \in Q_1$ , we denote by  $t(e) \in Q_0$  (resp.  $h(e) \in Q_0$ ) the vertex at which  $e$  starts (resp. ends). Take a dimension vector  $\mathbf{v} = (v_i) \in (\mathbb{Z}_{\geq 0})^{Q_0}$  and put  $V_i = \mathbb{C}^{v_i}$ . We define

$$\mathbf{M}(Q; \mathbf{v}) := \bigoplus_{e \in Q_1} \text{Hom}(V_{t(e)}, V_{h(e)})$$

and

$$G(\mathbf{v}) := \prod_{i \in Q_0} \text{GL}(V_i).$$

Note that  $G(\mathbf{v})$  naturally acts on  $\mathbf{M}(Q; \mathbf{v})$  and the quotient gives the moduli stack of representations of  $Q$  with dimension vectors  $\mathbf{v}$ . Let  $\chi_Q: \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$  be the bilinear form <sup>5</sup> given by

$$\chi_Q(\mathbf{v}, \mathbf{v}') := - \sum_{i,j \in Q_0} Q_{ij} v_i v'_j + \sum_{i \in Q_0} v_i v'_i.$$

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<sup>5</sup>This is the Euler form on the Grothendieck group of the category of finite-dimensional representations of  $Q$ .

Then we have

$$\dim M(Q; \mathbf{v}) - \dim G(\mathbf{v}) = -\chi_Q(\mathbf{v}, \mathbf{v}).$$

Let  $W$  be a potential, that is, a finite linear combination of cyclic paths in  $Q$ . Let  $f_{W, \mathbf{v}}$  be the  $G(\mathbf{v})$ -invariant function on  $M(Q; \mathbf{v})$  defined by taking the trace of the map associated to the potential  $W$ . A point in the critical locus  $\text{crit}(f_{W, \mathbf{v}})$  gives a  $J(Q, W)$ -module and the quotient stack

$$[\text{crit}(f_{W, \mathbf{v}})/G(\mathbf{v})]$$

gives the moduli stack of  $J(Q, W)$ -modules with dimension vectors  $\mathbf{v}$ .<sup>6</sup>

**Definition 1.8.** For  $(Q, W)$  and  $\mathbf{v}$ , we define motivic Donaldson-Thomas invariant by

$$\mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v}) := \frac{[\text{crit}(f_{W, \mathbf{v}})]_{\text{vir}}}{[G(\mathbf{v})]_{\text{vir}}} \in \widetilde{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}.$$

## 2 Cut of a QP and truncated Jacobian

### 2.1 Cut of a QP

Let  $(Q, W)$  be a QP. To each subset  $C \subset Q_1$  we associate a grading  $g_C$  on  $Q$  by

$$g_C(a) = \begin{cases} 1 & a \in C, \\ 0 & a \notin C. \end{cases}$$

Denote by  $Q_C$ , the subquiver of  $Q$  with the vertex set  $Q_0$  and the arrow set  $Q_1 \setminus C$ .

**Definition 2.1** ([HI, §3]). A subset  $C \subset Q_1$  is called a cut if  $W$  is homogeneous of degree 1 with respect to  $g_C$ .

If  $C$  is a cut, then  $g_C$  induces a grading on  $J(Q, W)$  as well. The degree 0 part of  $J(Q, W)$  is denoted by  $J(Q, W)_C$  and called the *truncated Jacobian algebra*. We have

$$\begin{aligned} J(Q, W)_C &= J(Q, W)/\langle C \rangle \\ &= \mathbb{C}Q_C / \langle \partial_a W \mid a \in C \rangle. \end{aligned}$$

Here we will show two examples of cuts.

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<sup>6</sup>In this sense, the function  $f_{W, \mathbf{v}}$  is called a Chern-Simons functional.

## 2.2 Example(1) : bipartite graph and perfect matching

The following example is studied by [IU].

Let  $\Sigma$  be a real 2-dimensional oriented manifold and  $\Gamma$  be a bipartite graph on  $\Sigma$ , that is,  $\Gamma$  is a triple  $(B, R, E)$  of disjoint finite subsets  $B$  (the set of blue vertices) and  $R$  (the set of red vertices) of  $\Sigma$  and a set of 1-cells  $E$  such that

- any two elements of  $E$  do not intersect in their interiors.
- each element of  $E$  connects one element in  $B$  and another element in  $R$ .

We take the dual graph of  $\Gamma$ . For each element  $e$  in  $E$ , we define the orientation of the dual edge  $\hat{e}$  so that  $\hat{e}$  crosses with  $e$  keeping the blue boundary of  $e$  on the right hand side. Let  $Q_\Gamma$  denote the resulting quiver. For  $b \in B$  (resp.  $r \in R$ ), let  $w_b$  (resp.  $w_r$ ) be the minimal cyclic path in  $Q_\Gamma$  which goes around  $b$  (resp.  $r$ ) clockwise (resp. anti-clockwise). We put

$$w_\Gamma := \sum_{b \in B} w_b - \sum_{r \in R} w_r.$$

**Example 2.2.** Let  $\Gamma$  be the bipartite graph on a torus in the left of Figure 2. The corresponding quiver  $Q_\Gamma$  is given in the right of Figure 2. The potential  $W_\Gamma$  is given by

$$W_\Gamma = a_1 b_1 c_1 d_1 - a_1 b_2 c_1 d_2 - a_2 b_1 c_2 d_1 + a_2 b_2 c_2 d_2.$$

The quiver with potential is known to be derived equivalent to the quotient stack  $[(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/(\mathbb{Z}/2\mathbb{Z})]$  where  $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{SL}(\mathbb{C}, 2)$  acts fiberwise.

A *perfect matching*  $P$  is a subset of  $E$  such that each element  $v \in B \cup R$  there exists exactly one element in  $P$  which has  $v$  as its boundary. It is easy to check that any perfect matching  $P$ , as a subset of  $(Q_\Gamma)_1 = E$ , gives a cut of the QP  $(Q_\Gamma, W_\Gamma)$ .

**Example 2.3.** Let  $P$  be the perfect matching in Figure 3. The corresponding cut is  $\{a_1, a_2\}$ .

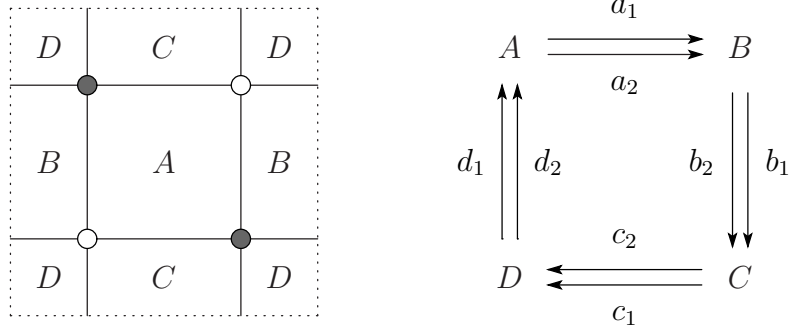


Figure 2: an example of a bipartite graph and the quiver

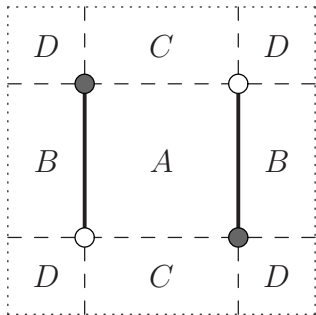


Figure 3: an example of a perfect matching

## 2.3 Example(2) Geometric helices on Del Pezzo surfaces

The following example is studied by [BS].

Let  $Y$  be a Del-Pezzo surface and let  $(E_i)_{i=1,\dots,n}$  be a full exceptional collection on  $Y$ . Put  $\mathbb{E} = \bigoplus_{i=1}^n E_i$  and define  $A(\mathbb{E}) := \text{End}(\mathbb{E})$ . We put

$$\mathbb{H}(\mathbb{E}) = (E_i)_{i \in \mathbb{Z}} := (\dots, \omega_Y \otimes E_n, E_1, \dots, E_n, \omega_Y^{-1} \otimes E_1, \dots).$$

and assume that

- $(E_i, \dots, E_{i+N-1})$  is an exceptional collection on  $Y$  for any  $i$ , and
- $\text{Hom}^k(E_i, E_j) = 0$  for any  $k \neq 0$  and any  $i < j$ .

Such a sequence  $(E_i)_{i \in \mathbb{Z}}$  is called a *geometric helix* ([BP94]). The *rolled up helix algebra* is the  $\mathbb{Z}$ -graded algebra

$$B(\mathbb{H}) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(E, \omega_Y^{-k} \otimes E)$$

with the obvious multiplication.

The following theorem is proved in [Kela, §6.9] and [dTdVdB, Appendix A].

**Theorem 2.4.** *There is a QP  $(Q, W)$  and a cut  $C$  such that*

$$B(\mathbb{H}) \simeq J(Q, W), \quad A(\mathbb{E}) \simeq J(Q, W)_C.$$

**Example 2.5.** *We take  $Y := \mathbb{P}_1 \times \mathbb{P}_1$  and a geometric helix*

$$\dots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1), \mathcal{O}(2, 2) = \omega_Y^{-1} \otimes \mathcal{O}, \dots$$

where we put  $\mathcal{O}(a, b) = \pi_1^*(\mathcal{O}_{\mathbb{P}_1}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}_1}(b))$ . We take the quiver  $Q$  in Figure 4 and the potential

$$W := \sum_{i,j \in \{1,2\}} U_{ij}(t_i s_j - S_j T_i),$$

then  $(Q, W)$  satisfies the conditions in Theorem 2.4.

Note that  $\mathbb{P}_1 \times \mathbb{P}_1$  gives a crepant resolution of the quotient singularity  $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/(\mathbb{Z}/2\mathbb{Z})$ . In fact, the QP in this example is obtained by mutating the one in Example 2.2 at the vertex  $D$  and they are derived equivalent.

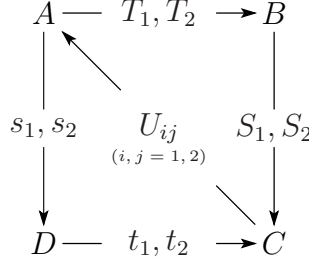


Figure 4: quiver for  $\mathbb{P}_1 \times \mathbb{P}_1$

## 2.4 Mutation of QP and cut

Extending Fomin and Zelevinsky mutations of quivers [FZ02], Derksen, Weyman, and Zelevinsky have introduced the notion of mutation of QPs in [DWZ08]. As pointed out in [AO], we can extend the definition to the graded setting. Let  $(Q, W, d)$  be a  $\mathbb{Z}$ -graded quiver with a homogeneous potential of degree  $r$  and  $k$  be vertex of  $Q$ . We define  $\tilde{\mu}_k^L(Q, W, d) = (\tilde{Q}, \tilde{W}, \tilde{d})$  the left mutation of  $(Q, W, d)$  at vertex  $k$  as follows :

- (1) the new quiver  $\tilde{Q}$  is defined as follows :
  - (a) for any subquiver  $u \xrightarrow{a} k \xrightarrow{b} v$  with  $k, u$  and  $v$  pairwise different vertices, we add an arrow  $[ba]: u \rightarrow v$ ;
  - (b) we replace all arrows  $a$  incident with  $k$  by an arrow  $a^*$  in the opposite direction.
- (2) The new potential  $\tilde{W}$  is defined by the sum  $[W] + \Delta$  where  $[W]$  is formed from the potential  $W$  replacing all compositions  $ba$  through the vertex  $k$  by the new arrows  $[ba]$ , and where  $\Delta$  is the sum  $\sum a^* b^* [ba]$ .
- (3) The new degree  $\tilde{d}$  is defined as follows :
  - (a)  $\tilde{d}(a) = d(a)$  for  $a$  not incident to  $k$  ;
  - (b)  $\tilde{d}([ba]) = d(b) + d(a)$  for a composition  $ba$  passing through  $k$  ;
  - (c)  $\tilde{d}(a^*) = -d(a) + r$  if  $t(a) = k$ ;
  - (d)  $\tilde{d}(b^*) = -d(b)$  if the source of  $s(b) = k$ .

By the graded version ([AO, Theorem 6.4]) of the splitting theorem [DWZ08, Theorem 4.6], any graded QP  $(Q, W, d)$  has a direct sum decomposition

$$(Q, W, d) = (Q, W, d)^{\text{red}} \oplus (Q, W, d)^{\text{triv}}$$

into a reduced graded QP and a trivial graded QP. The decomposition is unique up to graded right equivalence.

Assume that the potential  $W$  is generic in the sense of [DWZ08]. Then the underlying quiver of the reduction  $\mu_k^L(Q, W, d) := (\tilde{Q}, \tilde{W}, \tilde{d})^{\text{red}}$  of the left mutation  $\tilde{\mu}_k^L(Q, W, d) = (\tilde{Q}, \tilde{W}, \tilde{d})$  coincides with Fomin-Zelevinsky's mutation.

**Definition 2.6** ([HI, Definition 6.12]). *Let  $C$  be a cut of a QP  $(Q, W)$ . We say that a vertex  $k$  of  $Q$  is a strict source (resp. sink) of  $(Q, C)$  if all arrows ending (resp. starting) at  $x$  belong to  $C$  and all arrows starting (ending) at  $x$  do not belong to  $C$ .*

**Example 2.7.** (1) *Let  $(Q, W)$  and  $C$  be given as in §2.2. A vertex is a strict source or a strict sink if and only if the vertices of the corresponding face of the bipartite graph is perfectly matched by the perfect matching.*

(2) *Let  $(Q, W)$  and  $C$  be given as in §2.3. Then the vertex corresponding to the exceptional object  $E_1$  is a strict source.*

The underlying graded quiver of  $\mu_k^L(Q, W, d_C)$  is given as follows:

- (a) add degree 1 arrows  $[ba]$  ;
- (b) replace  $a$  with a degree 0 arrow  $a^*$  ;
- (c) cancel 2-cycles<sup>7</sup>.

The new degree gives a cut of the mutated QP  $\mu_k(Q, W)$ . Let  $\mu_k C$  denote this cut.

**Remark 2.8.** *Given a strict source  $k$ , a new cut  $C_k$  of  $(Q, W)$  is defined ([HI, Definition 6.10])<sup>8</sup>. If  $(Q, W)$  is Calabi-Yau, then  $J(\mu_k Q, \mu_k W)_{\mu_k C}$  is isomorphic to  $J(Q, W)_{C_k}$ .*

### 3 Factorization property

Let  $(Q, W)$  be a QP and  $C$  be a cut. The grading  $g_C$  gives a  $\mathbb{C}^*$ -action on  $M(Q; \mathbf{v})$  and the action satisfies the assumption of Theorem 1.4 if we put

$$X = M(Q; \mathbf{v}), \quad T = \mathbb{C}^*, \quad f = f_{W, \mathbf{v}}.$$

---

<sup>7</sup>Since we assume  $W$  is generic, we can see any 2-cycle has degree 1. So this step has no ambiguity even in the graded sense.

<sup>8</sup>They call  $C_k$  the *cut mutation*.

Hence we have

$$\mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v}) \in \widetilde{\mathcal{M}}_{\mathbb{C}} \quad (4)$$

and

$$\begin{aligned} \mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v}) &= \frac{\mathbb{L}^{-\dim M(Q; \mathbf{v})/2} \cdot (f_{W, \mathbf{v}}^{-1}(1) - f_{W, \mathbf{v}}^{-1}(0))}{[G(\mathbf{v})]_{\text{vir}}} \\ &= \mathbb{L}^{\chi_Q(\mathbf{v}, \mathbf{v})/2} \times \frac{f_{W, \mathbf{v}}^{-1}(1) - f_{W, \mathbf{v}}^{-1}(0)}{[G(\mathbf{v})]}. \end{aligned}$$

We define the *refined DT invariant* by

$$m_{\text{ref}}(Q, W; \mathbf{v}) := \mathbb{W}(\mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v})) \in \mathbb{Q}(q^{1/2}).$$

Throughout this section, we will use simplified notations such as  $M(\mathbf{v})$  and  $f_{\mathbf{v}}$  instead of  $M(Q; \mathbf{v})$  and  $f_{W, \mathbf{v}}$  omitting  $Q$  and  $W$ .

### 3.1 Filtration and motivic invariants

The next proposition directly follows Theorem 1.4 and Definition 1.6.

**Proposition 3.1.** *Let  $X$ ,  $T$ ,  $f$  be as in Theorem 1.4 and  $Y$  be a smooth  $T$ -invariant closed subset  $X$  with dimension  $d$ . We assume that the  $T$ -action on  $Y$  and  $f|_Y$  satisfies the conditions in Theorem 1.4 as well. Then we have*

$$[\text{crit}(f)]_{\text{vir}} = [\text{crit}(f|_{X \setminus Y})]_{\text{vir}} + \mathbb{L}^{-\frac{d}{2}} [\text{crit}(f|_Y)]_{\text{vir}}.$$

**Corollary 3.2.** *Let*

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = X$$

*be a filtration of  $X$  by  $T$ -invariant open subsets. We put  $X_{\alpha} := U_{\alpha} \setminus U_{\alpha-1}$  and  $f_{\alpha} := f|_{X_{\alpha}}$ . Assume that  $X_{\alpha}$ ,  $f_{\alpha}$  and the  $T$ -action satisfies the conditions in Theorem 1.4. Then we have*

$$[\text{crit}(f)]_{\text{vir}} = \sum_{\alpha=1}^n \mathbb{L}^{\frac{-\dim X + \dim X_{\alpha}}{2}} [\text{crit}(f_{\alpha})]_{\text{vir}}.$$

### 3.2 Filtration by HN property

In this subsection, we repeat [KSa, §5.2] to fix the notations. We put

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{im} z > 0 \text{ or } z \in \mathbb{R}_{>0}\}$$



and define the total order  $\succ$  on  $\mathbb{H}$  by

$$z_1 \succ z_2 \stackrel{\text{def}}{\iff} \text{Arg}(z_1) > \text{Arg}(z_2) \text{ or } \{\text{Arg}(z_1) = \text{Arg}(z_2), |z_1| > |z_2|\}.$$

We identify the Grothendieck group  $K_0(\text{mod}(J(Q, W)))$  with  $\mathbb{Z}^{Q_0}$  and put  $N := (\mathbb{Z}_{\geq 0})^{Q_0}$ . Let

$$Z: \mathbb{Z}^{Q_0} \rightarrow \mathbb{C}$$

be a central charge, that is, a group homomorphism such that

$$Z(N \setminus \{0\}) \subset \mathbb{H}.$$

Let  $M_{Z\text{-ss}}(\mathbf{v})$  denote the open subset of  $M(\mathbf{v})$  consisting of  $Z$ -semistable  $Q$ -modules.

**Definition 3.3.** For  $\mathbf{v} \in N$ , we define the finite set  $P_Z(\mathbf{v})$  by

$$\left\{ \mathbf{v}_\bullet = (\mathbf{v}_i) \in N^n \mid n \geq 1, \sum \mathbf{v}_i = \mathbf{v}, \text{Arg} Z(\mathbf{v}_1) > \dots > \text{Arg} Z(\mathbf{v}_n) \right\}.$$

We introduce a partial order on  $P_Z(\mathbf{v})$  by

$$(\mathbf{v}_1, \dots, \mathbf{v}_n) \underset{Z}{\leq} (\mathbf{v}'_1, \dots, \mathbf{v}'_{n'}) \\ \stackrel{\text{def}}{\iff} \mathbf{v}_1 = \mathbf{v}'_1, \dots, \mathbf{v}_{i-1} = \mathbf{v}'_{i-1} \text{ and } Z(\mathbf{v}_i) \prec Z(\mathbf{v}'_{i'}) \text{ for some } i.$$

**Definition 3.4.** Let denote by  $M(\mathbf{v}; \mathbf{v}_\bullet)$  the subset of  $M(\mathbf{v})$  consisting of  $Q$ -modules which admit increasing filtrations

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$$

such that

$$\dim(E_i/E_{i-1}) = \mathbf{v}_i$$

for any  $i = 1, \dots, n$ .

**Lemma 3.5.** The subset  $M(\mathbf{v}; \mathbf{v}_\bullet) \subset M(\mathbf{v})$  is closed.

*Proof.* We put

$$\text{Fl}(\mathbf{v}_\bullet) := \prod_{i \in Q_0} \text{Fl}(v_{1,i}, \dots, v_{n,i})$$

where  $\text{Fl}(v_{1,i}, \dots, v_{n,i})$  is the flag varieties of all flags in  $V_i$  of type  $(v_{1,i}, \dots, v_{n,i})$ . Note that the following subset of  $M(\mathbf{v}) \times \text{Fl}(\mathbf{v}_\bullet)$  is closed :

$$\{(X, F) \in M(\mathbf{v}) \times \text{Fl}(\mathbf{v}_\bullet) \mid F \text{ is } X\text{-stable}\}.$$

Then  $M(\mathbf{v}; \mathbf{v}_\bullet)$  is closed since it is the image of the closed set above under the projection

$$M(\mathbf{v}) \times \text{Fl}(\mathbf{v}_\bullet) \rightarrow M(\mathbf{v})$$

which is proper. □

**Definition 3.6.** We define the locally closed subset  $M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet)$  of  $M(\mathbf{v})$  by

$$M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet) := M(\mathbf{v}; \mathbf{v}_\bullet) - \bigcup_{\mathbf{v}'_\bullet \leq \mathbf{v}_\bullet} M(\mathbf{v}; \mathbf{v}'_\bullet).$$

**Lemma 3.7.** (1) A  $\mathbb{C}$ -point in  $M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet)$  represents a  $Q$ -module whose HN filtration is of type  $\mathbf{v}_\bullet$ .

(2)  $M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet)$  is smooth and

$$\text{codim } M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet) = - \sum_{a < b} \chi_Q(\mathbf{v}_a, \mathbf{v}_b).$$

*Proof.* (1) See [KSa, pp49, Lemma 2].

(2) Fix a direct sum decompositions  $V_i = \bigoplus_{a=1}^n V_{a,i}$  with  $V_{a,i} \simeq \mathbb{C}^{v_{a,i}}$ . We define the subspace  $M(\mathbf{v}_\bullet)$  of  $M(\mathbf{v})$  by

$$M(\mathbf{v}_\bullet) := \bigoplus_{a \geq b, e \in Q_1} \text{Hom}(V_{a,t(e)}, V_{b,h(e)}).$$

We put

$$M_{Z\text{-ss}}(\mathbf{v}_\bullet) := \pi^{-1}(M_{Z\text{-ss}}(\mathbf{v}_1) \times \cdots \times M_{Z\text{-ss}}(\mathbf{v}_n))$$

where

$$\pi: M(\mathbf{v}_\bullet) \rightarrow M(\mathbf{v}_1) \times \cdots \times M(\mathbf{v}_n)$$

is the natural projection. Note that  $\pi$  is a trivial vector bundle and so  $M_{Z\text{-ss}}(\mathbf{v}_\bullet)$  is smooth.

Let

$$\text{HN}: M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet) \rightarrow \text{FL}(\mathbf{v}_\bullet)$$

be the map defined by taking the Harder-Narashimhan filtration. Then HN is a Zariski locally trivial fibration whose fibers are isomorphic to  $M_{Z\text{-ss}}(Q; \mathbf{v}_\bullet)$ . So  $M_{Z\text{-HN}}(Q; \mathbf{v}, \mathbf{v}_\bullet)$  is smooth.

The computation of the codimension is straightforward.  $\square$

Let  $f_{\mathbf{v}}^{Z\text{-ss}}$  (resp.  $f_{\mathbf{v}_\bullet}^{Z\text{-ss}}, f_{\mathbf{v}_\bullet}^{Z\text{-HN}}$ ) denote the restriction of the Chern-Simons functional  $f_{\mathbf{v}} = f_{W,\mathbf{v}}$  on  $M_{Z\text{-ss}}(\mathbf{v})$  (resp.  $M_{Z\text{-ss}}(\mathbf{v}_\bullet), M_{Z\text{-HN}}(\mathbf{v}, \mathbf{v}_\bullet)$ ).

**Proposition 3.8** (see [KSa, pp51 Theorem 5]). Assume that the  $QP$  has a cut, then we have

$$\frac{[\text{crit}(f_{\mathbf{v}_\bullet}^{Z\text{-HN}})]_{\text{vir}}}{[G(\mathbf{v})]_{\text{vir}}} = \mathbb{L}^{-\sum_{a > b} \chi_Q(\mathbf{v}_a, \mathbf{v}_b)} \times \prod_a \frac{[\text{crit}(f_{\mathbf{v}_a}^{Z\text{-ss}})]_{\text{vir}}}{[G(\mathbf{v}_a)]_{\text{vir}}}.$$

*Proof.* The claim is a consequence of the following two identities, which is obtained from the descriptions of in the proof of Lemma 3.7 :

$$\begin{aligned} [\text{crit}(f_{\mathbf{v}_\bullet}^{Z\text{-HN}})]_{\text{vir}} &= [\text{crit}(f_{\mathbf{v}_\bullet}^{Z\text{-ss}})]_{\text{vir}} \times [\text{FL}(\mathbf{v}_\bullet)]_{\text{vir}} \\ &= [\text{crit}(f_{\mathbf{v}_\bullet}^{Z\text{-ss}})]_{\text{vir}} \times \frac{[\text{G}(\mathbf{v})]_{\text{vir}}}{\prod_a [\text{G}(\mathbf{v}_a)]_{\text{vir}} \times [\mathbb{L}^{\sum_{i,a>b} v_{a,i} v_{b,i}}]_{\text{vir}}} \end{aligned}$$

and

$$[\text{crit}(f_{\mathbf{v}_\bullet}^{Z\text{-ss}})]_{\text{vir}} = \left[ \mathbb{L}^{\sum_{i,a>b} Q_{ij} v_{a,i} v_{b,i}} \right]_{\text{vir}} \times \prod_a [\text{crit}(f_{\mathbf{v}_a}^{Z\text{-ss}})]_{\text{vir}}.$$

For the second identity, we use the motivic Thom-Sebastiani theorem (Theorem 1.3).  $\square$

Let  $\langle \bullet, \bullet \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$  be the skewsymmetric bilinear form given by

$$\langle \mathbf{v}, \mathbf{v}' \rangle := \chi_Q(\mathbf{v}, \mathbf{v}') - \chi_Q(\mathbf{v}', \mathbf{v}).$$

Combining the results in this subsection, we get the following theorem :

**Theorem 3.9.** *Assume that the QP has a cut, then we have*

$$\mathfrak{M}_{\text{vir}}(Q, W, \mathbf{v}) := \sum_{\mathbf{v}_\bullet \in P_Z(\mathbf{v})} \left( \mathbb{L}^{\frac{1}{2} \sum_a \langle \mathbf{v}_a, \mathbf{v}_b \rangle} \times \prod_a \mathfrak{M}_{\text{vir}}(Q, W, \mathbf{v}_a) \right)$$

### 3.3 Factorization property

We assume that the QP has a cut.

**Definition 3.10.** *The motivic torus associated to  $Q$  is*

$$\hat{\mathcal{T}}_Q := \prod_{\mathbf{v} \in N} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y_{\mathbf{v}}$$

where  $y_{\mathbf{v}}$  's are formal variables which satisfy the relation

$$y_{\mathbf{v}_1} \cdot y_{\mathbf{v}_2} = \mathbb{L}^{\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{2}} y_{\mathbf{v}_1 + \mathbf{v}_2}.$$

**Definition 3.11.** *We define the generating series of the motivic Donaldson-Thomas invariants of  $(Q, W)$  by*

$$\mathcal{A} = \mathcal{A}_{Q,W} := 1 + \sum_{\mathbf{v} \in N} \mathfrak{M}_{\text{vir}}(Q, W, \mathbf{v}) \cdot y_{\mathbf{v}} \in \hat{\mathcal{T}}_Q.$$

**Definition 3.12.** Let  $l \subset \mathbb{H}$  be a ray and  $Z$  be a central charge. We put

$$\mathcal{A}^{Z,l} := 1 + \sum_{Z(\mathbf{v}) \in l} \frac{[\text{crit}(f_{\mathbf{v}}^{Z\text{-ss}})]_{\text{vir}}}{[G(\mathbf{v})]} \cdot y_{\mathbf{v}} \in \hat{\mathcal{T}}_Q.$$

Theorem 3.9 implies the following factorization formula :

**Theorem 3.13.** Assume that the QP has a cut, then we have

$$\mathcal{A} = \prod_l^{\curvearrowright} \mathcal{A}^{Z,l}$$

where the product is taken in the clockwise order over all rays.

**Definition 3.14.** The quantum torus associated to  $Q$  is

$$\hat{T}_Q := \prod_{\mathbf{v} \in N} \mathbb{Q}(q^{1/2}) \cdot y_{\mathbf{v}}$$

where  $y_{\mathbf{v}}$  's are formal variables which satisfy the relation

$$y_{\mathbf{v}_1} \cdot y_{\mathbf{v}_2} = q^{\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{2}} y_{\mathbf{v}_1 + \mathbf{v}_2}.$$

**Definition 3.15.** We define the generating series of the refined Donaldson-Thomas invariants of  $(Q, W)$  by

$$A = A_{Q,W} := m_{\text{ref}}(Q, W, \mathbf{v}) \cdot y_{\mathbf{v}} \in \hat{T}_Q.$$

In the same way, we define  $A^{Z,l}$ . Since  $\mathbb{W}$  is a ring homomorphism, we get the following factorization formula for refined DT invariants :

**Corollary 3.16.** Assume that the QP has a cut, then we have

$$A = \prod_l^{\curvearrowright} A^{Z,l}$$

where the product is taken in the clockwise order over all rays.

## 4 Wall-crossing formula

### 4.1 Motives for $J(Q, W)_C$

We put

$$\begin{aligned} \chi_C(\mathbf{v}, \mathbf{v}') &:= \sum_{c \in C} v_{t(c)} v'_{h(c)}, \\ \chi_{Q_C}(\mathbf{v}, \mathbf{v}') &:= \chi_Q(\mathbf{v}, \mathbf{v}') - \chi_C(\mathbf{v}, \mathbf{v}'). \end{aligned}$$

We put

$$d := \dim M(Q; \mathbf{v}) - \dim M(Q_C; \mathbf{v}) = \chi_C(\mathbf{v}, \mathbf{v}).$$

Let  $M(Q, W, C; \mathbf{v})$  be the subset of  $M(Q_C; \mathbf{v})$  consisting of  $J(Q, W)_C$ -modules. The following theorem is a generalization of [BBS, Equation (2.4)] and [Hua, Theorem 7.3].

**Theorem 4.1.**

$$[\varphi_{f_{W, \mathbf{v}}}] = \mathbb{L}^d \cdot [M(Q, W, C; \mathbf{v})].$$

*Proof.* Note that we have

$$\begin{aligned} [\varphi_{f_{W, \mathbf{v}}}] &= f_{W, \mathbf{v}}^{-1}(1) - f_{W, \mathbf{v}}^{-1}(0) \\ &= \frac{[M(Q; \mathbf{v})] - f_{W, \mathbf{v}}^{-1}(0)}{\mathbb{L} - 1} - f_{W, \mathbf{v}}^{-1}(0) \\ &= \frac{[M(Q; \mathbf{v})] - \mathbb{L} \cdot f_{W, \mathbf{v}}^{-1}(0)}{\mathbb{L} - 1}. \end{aligned} \tag{5}$$

Let  $\pi: M(Q; \mathbf{v}) \rightarrow M(Q_C; \mathbf{v})$  be the natural projection. This is a trivial vector bundle of rank  $d$ . Since we have

$$W = \sum_{e \in C} \partial_e W,$$

the restriction of  $f_{W, \mathbf{v}}$  to the fiber  $\pi^{-1}(x)$  is zero if  $x \in M(Q, W, C; \mathbf{v})$  and is a non-zero linear function if  $x \notin M(Q, W, C; \mathbf{v})$ . Hence we have

$$f_{W, \mathbf{v}}^{-1}(0) = \mathbb{L}^d \cdot [M(Q, W, C; \mathbf{v})] + \mathbb{L}^{d-1}([M(Q_C; \mathbf{v})] - [M(Q, W, C; \mathbf{v})]). \tag{6}$$

Substitute (6) to (5), then the claim follows.  $\square$

**Remark 4.2.** For the cohomological Hall algebra, a similar statement is proved in [KSa, Proposition 6].

## 4.2 Mutation and bilinear forms

Let  $(Q', W', C')$  be the new QP with the cut given by the mutation at a strict source  $k$  of  $(Q, W, C)$ .

We identify the Grothendieck group  $K_0(\text{mod}(J(Q, W)))$  with  $\mathbb{Z}^{Q_0}$  as before. We define

$$\phi_k: \mathbb{Z}^{Q_0} \xrightarrow{\sim} \mathbb{Z}^{Q'_0}$$

by

$$\phi_k([s_i]) = \begin{cases} [s'_i] & i \neq k, \\ -[s'_k] + \sum_{t(b)=k} [s'_{h(b)}] & i = k. \end{cases}$$

Then we can verify the following :

**Lemma 4.3.**

$$\begin{aligned}\chi_Q(\mathbf{v}, \mathbf{w}) &= \chi_{Q'}(\phi_k(\mathbf{v}), \phi_k(\mathbf{w})), \\ \chi_C(\mathbf{v}, \mathbf{w}) &= \chi_{C'}(\phi_k(\mathbf{v}), \phi_k(\mathbf{w})), \\ \chi_{Q_C}(\mathbf{v}, \mathbf{w}) &= \chi_{Q_{C'}}(\phi_k(\mathbf{v}), \phi_k(\mathbf{w})).\end{aligned}$$

**Remark 4.4.** *The bilinear form  $\chi_Q$  is the Euler form of the derived category of the Ginzburg's dg algebra. The map  $\phi_k$  is induced from the Keller-Yang's derived equivalence ([KY, Kela]). This is the origin of the first equation.*

*If the Ginzburg's dg algebra is concentrated on degree 0, then the bilinear form  $\chi_{Q_C}$  is the Euler form of the derived category of  $J(Q, W)_C$ . In such a cases, we have the derived equivalence between  $J(Q, W)_C$  and  $J(Q', W')_{C'}$  which induces  $\phi_k$ . This is the origin of the third equation.*

### 4.3 Mutation and a motivic identity

For a strict source  $k$ , we define open subsets  $M(Q; \mathbf{v})_k \subset M(Q; \mathbf{v})$  and  $M(Q_C; \mathbf{v})_k \subset M(Q_C; \mathbf{v})$  by

$$\begin{aligned}M(Q; \mathbf{v})_k &:= \{V \in M(Q; \mathbf{v}) \mid \text{Hom}(s_k, V) = 0\}, \\ M(Q_C; \mathbf{v})_k &:= \{V \in M(Q_C; \mathbf{v}) \mid \text{Hom}(s_k, V) = 0\}\end{aligned}$$

and put

$$M(Q, W, C; \mathbf{v})_k := M(Q_C; \mathbf{v})_k \cap M(Q, W, C; \mathbf{v}).$$

Note that we have  $\pi^{-1}(M(Q_C; \mathbf{v})_k) = M(Q; \mathbf{v})_k$ . We put

$$f_{W, \mathbf{v}, k} := f_{W, \mathbf{v}}|_{M(Q, W, C, \mathbf{v})_k}.$$

We can prove the following in the same way as Theorem 4.1 :

**Proposition 4.5.**

$$[\varphi_{f_{W, \mathbf{v}, k}}] = \mathbb{L}^d \cdot [M(Q, W, C; \mathbf{v})_k].$$

We use the upper subscript  $M(\dots)^k$  for the ones with the condition  $\text{Hom}(-, s'_k) = 0$ .

**Proposition 4.6.**

$$\frac{[M(Q, W, C; \mathbf{v})_k]}{[G(\mathbf{v})]} = \frac{[M(Q', W', C'; \mathbf{v}')^k]}{[G(\mathbf{v}')]}\tag{7}$$

where  $\mathbf{v}' = \phi_k(\mathbf{v})$ .

*Proof.* This is a consequence of Proposition 4.12 and Lemma 1.1. <sup>9</sup>  $\square$

Combining this with Proposition 4.5 and Lemma 4.3, we get the following identity of the virtual motives of the same moduli stack with different Chern-Simons functionals :

**Theorem 4.7.**

$$\mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v})_k = \mathfrak{M}_{\text{vir}}(Q', W'; \mathbf{v}')^k.$$

#### 4.4 Wall-crossing formula for motivic DT invariants

In this subsection, we will work over  $\mathbb{Q}$  and use the notations as  $\hat{\mathcal{T}}_Q^{\mathbb{Q}} := \hat{\mathcal{T}}_Q \otimes \mathbb{Q}$ .

We put  $N_{Q,Q'} := \phi_k^{-1}(N) \cap N$  and

$$\hat{\mathcal{T}}_{Q,Q'}^{\mathbb{Q}} := \prod_{\mathbf{v} \in N_{Q,Q'}} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y_{\mathbf{v}} \subset \hat{\mathcal{T}}_Q^{\mathbb{Q}}.$$

Note that  $\hat{\mathcal{T}}_{Q,Q'}^{\mathbb{Q}}$  is also a subalgebra of  $\hat{\mathcal{T}}_{Q'}^{\mathbb{Q}}$ . We put

$$\begin{aligned} \mathcal{A}_{Q,W,k} &:= 1 + \sum_{\mathbf{v} \in N_{Q,Q'}} \mathfrak{M}_{\text{vir}}(Q, W; \mathbf{v})_k \cdot y_{\mathbf{v}} \\ &= 1 + \sum_{\mathbf{v} \in N_{Q,Q'}} \mathfrak{M}_{\text{vir}}(Q', W'; \mathbf{v}')^k \cdot y_{\mathbf{v}} \quad (\text{Theorem 4.7}) \\ &\in \hat{\mathcal{T}}_{Q,Q'}^{\mathbb{Q}}. \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(s_k) &:= \sum_{n \geq 0} \frac{[\text{pt}]}{[\text{GL}_n]_{\text{vir}}} \cdot y_{[(s_k)^{\oplus n}]} \in \mathcal{T}_Q^{\mathbb{Q}}, \\ \mathbb{E}(s'_k) &:= \sum_{n \geq 0} \frac{[\text{pt}]}{[\text{GL}_n]_{\text{vir}}} \cdot y_{[(s'_k)^{\oplus n}]} \in \mathcal{T}_{Q'}^{\mathbb{Q}} \end{aligned}$$

(see Example 1.2). In the same way as Theorem 3.13, we can see the following factorizations :

$$\begin{aligned} \mathcal{A}_{Q,W} &= \mathcal{A}_{Q,W,k} \times \mathbb{E}(s_k) \in \hat{\mathcal{T}}_Q^{\mathbb{Q}}, \\ \mathcal{A}_{Q',W'} &= \mathbb{E}(s'_k) \times \mathcal{A}_{Q,W,k} \in \hat{\mathcal{T}}_{Q'}^{\mathbb{Q}}. \end{aligned}$$

Now we get the following wall-crossing formula for the motivic DT invariants :

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<sup>9</sup>In fact, the equivalence in Corollary 4.12 gives only the bijection of  $\mathbb{C}$ -valued points. But all the arguments in §4.6 can be applied for families of representations and we can see the isomorphism of the moduli stacks.

**Theorem 4.8.** *We have*

$$\mathcal{A}_{Q,W} \times \mathbb{E}(s_k)^{-1}, \quad \mathbb{E}(s'_k)^{-1} \times \mathcal{A}_{Q',W'} \in \hat{\mathcal{T}}_{Q,Q'}^{\mathbb{Q}}$$

*and they coincide.*

## 4.5 Refined DT invariants

We define  $\hat{T}_{Q,Q'}$  in the same way.

Taking the weight polynomial  $\mathbb{W}$  of the equation in Theorem 4.8, we get the following formula which describes the relation between the refined DT invariants of  $(Q, W)$  and  $(Q', W')$  in terms of the quantum dilogarithm :

**Theorem 4.9.** *We have*

$$A_{Q,W} \times \mathbb{E}_q(y_k)^{-1}, \quad \mathbb{E}_q(y_k^{-1})^{-1} \times A_{Q',W'} \in \hat{T}_{Q,Q'}$$

*and they coincide.*

## 4.6 Appendix : reminders on [DWZ08]

For a QP  $(Q, W)$  and a vertex  $k$ , we associate a new QP  $\tilde{\mu}_k(Q, W) = (\tilde{Q}, \tilde{W})$  as follows. We put  $\tilde{Q}_0 = Q_0$  and  $\tilde{Q}_1$  is the union of

- all the arrows  $c \in Q_1$  not incident to  $k$ ,
- a “composite” arrow  $[ba]$  from  $t(a)$  to  $h(b)$  for each  $a$  and  $b$  with  $h(a) = t(b) = k$ , and
- an opposite arrow  $a^*$  (resp.  $b^*$ ) for each incoming arrow  $a$  (resp. outgoing arrow  $b$ ) at  $k$ .

The new potential is given by

$$\tilde{W} := [W] + \Delta$$

where

$$\Delta := \sum_{a,b \in Q_1; h(a)=t(b)=k} [ba]a^*b^*$$

and  $[W]$  is obtained by substituting  $[ba]$  for each factor  $ba$  occurring in the expansion of  $W$ . What we have been assuming is that there is an automorphism  $\psi$  of  $\mathbb{C}\tilde{Q}$  such that we have a decomposition

$$(\psi(\tilde{Q}), \psi(\tilde{W})) \simeq (\psi(\tilde{Q})_{\text{red}}, \psi(\tilde{W})_{\text{red}}) \oplus (\psi(\tilde{Q})_{\text{triv}}, \psi(\tilde{W})_{\text{triv}}) \quad (8)$$



with  $\psi(\tilde{Q})_{\text{red}} = \mu_k(Q)$ . We put  $\mu_k(W) := \psi(\tilde{W})_{\text{red}}$ . The reader may refer [DWZ08, §5] for the details<sup>10</sup>.

We fix the automorphism  $\psi$  and the decomposition (8). Then a  $J_{\tilde{Q}, \tilde{W}}$ -module is canonically identified with a  $J_{\mu_k(Q, W)}$ -module.

Take  $V \in M(Q; \mathbf{v})_k$ . We define  $\tilde{V} := \oplus \tilde{V}_i$  by  $\tilde{V}_i := V_i$  for  $i \neq k$  and

$$\tilde{V}_k := \text{coker} \left( \sum_{t(b)=k} b: V_k \rightarrow \bigoplus_{t(b)=k} V_{h(b)} \right).$$

Note that the sum of the maps above is injective. We define the action of  $\mathbb{C}\tilde{Q}$  on  $\tilde{V}$  as follows:

- for an arrow  $c \in Q_1$  not incident to  $k$ , we associate

$$\tilde{V}_{t(c)} = V_{t(c)} \xrightarrow{c} V_{h(c)} = \tilde{V}_{h(c)},$$

- for a “composite” arrow  $[ba]$ , we associate the composition

$$\tilde{V}_{t([ba])} = V_{t(a)} \xrightarrow{ba} V_{h(b)} = \tilde{V}_{h([ba])},$$

- for an opposite arrow  $a^*$  of an incoming arrow  $a$  at  $k$ , we associate the map induced by

$$\sum_{t(b)=k} \partial_{[ba]} W: \bigoplus_{t(b)=k} V_{h(b)} \longrightarrow V_{t(a)} = \tilde{V}_{h(a^*)},$$

- for an opposite arrow  $b^*$  of an outgoing arrow  $b$  at  $k$ , we associate

$$\tilde{V}_{t(b^*)} = V_{h(b)} \hookrightarrow \bigoplus_{t(b')=k} V_{h(b')} \twoheadrightarrow \tilde{V}_k = \tilde{V}_{h(b^*)}.$$

Then  $\tilde{V}$  belongs to  $M(\tilde{Q}, \mathbf{v}')^k$  where  $\mathbf{v}' = \phi_k(\mathbf{v})$ . We can verify that if  $V \in M(Q, W, \mathbf{v})_k$  then  $\tilde{V} \in M(\tilde{Q}, \tilde{W}, \mathbf{v}')^k$ .

Take  $U \in M(\tilde{Q}; \mathbf{u})^k$ . We define  $\hat{U} := \oplus \hat{U}_i$  by  $\hat{U}_i := U_i$  for  $i \neq k$  and

$$\hat{U}_k := \ker \left( \sum_{h(b^*)=k} b^*: \bigoplus_{h(b^*)=k} U_{t(b^*)} \rightarrow U_k \right).$$

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<sup>10</sup>In [DWZ08], it is shown that for a generic  $W$  we always have such an automorphism of the completion of  $\mathbb{C}\tilde{Q}$ . Here we assume that we have an automorphism of  $\mathbb{C}\tilde{Q}$ , otherwise the mutation of the potential can be infinite.

Note that the sum of the maps above is surjective. We define the action of  $\mathbb{C}\tilde{\tilde{Q}}$  on  $\hat{U}$  in a similar way. If  $U \in M(\tilde{\tilde{Q}}, \tilde{\tilde{W}}, \mathbf{u})^k$  then  $\tilde{U} \in M(\tilde{\tilde{Q}}, \tilde{\tilde{W}}, \phi_k^{-1}(\mathbf{u}))_k$ .

In the proof of [DWZ08, Theorem 5.7], an explicit equivalence between  $(Q, W)$  and  $(\tilde{\tilde{Q}}, \tilde{\tilde{W}})$  is given. This induces an identification of elements in  $M(\tilde{\tilde{Q}}, \tilde{\tilde{W}}, \mathbf{u})_k$  with ones in  $M(Q, W, \mathbf{u})_k$ .

We can verify the composition of these three functors is identity. Combined with the identification of  $J_{\tilde{\tilde{Q}}, \tilde{\tilde{W}}}$ -modules and  $J_{\mu_k(Q, W)}$ -modules we get the following equivalence:

**Proposition 4.10.**

$$\Phi: \text{mod}(J(Q, W))_k \xrightarrow{\sim} \text{mod}(J(Q', W'))^k. \quad (9)$$

**Remark 4.11.** *The derived equivalence given by [KY] induces the equivalence of the two categories above. Here we use [DWZ08]'s construction since we need the explicit description of  $\Phi$  to prove the following proposition.*

**Proposition 4.12.** *The equivalence (9) induces*

$$\text{mod}(J(Q, W)_C)_k \simeq \text{mod}(J(Q', W')_{C'})^k.$$

*Proof.* First, take  $V \in \text{mod}(J(Q, W)_C)_k$ . For each composite arrow  $[ba]$ , the map  $[ba]$  vanishes on  $\Phi(V)$  since the map  $a$  vanishes on  $V$ . Hence we have  $\Phi(V) \in \text{mod}(J(Q', W')_{C'})^k$ .

Next, assume that  $\Phi(V) \in \text{mod}(J(Q', W')_{C'})^k$  for  $V \in \text{mod}(J(Q, W))_k$ . For any  $a$  with  $h(a) = k$ ,  $a$  vanishes on  $V$  since  $\sum_{t(b)=k} [ba]$  vanished and  $\sum_{t(b)=k} b$  is injective. Hence we have  $V \in \text{mod}(J(Q, W)_C)_k$ .  $\square$

**Remark 4.13.** *This gives a generalization of a part of the results of [BS].*

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